

On the Real-rootedness of the Local h -polynomials of Edgewise Subdivisions of Simplexes

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Abstract. Athanasiadis conjectured, for every positive integer r , the local h -polynomial of r th edgewise subdivision of any abstract complex has only real zeros. In this paper, we prove this conjecture by the method of interlacing polynomials, which recently has been widely developed.

AMS Classification 2010: 26C10, 05E45, 52B45, 05A15

Keywords: real-rootedness, interlacing, local h -polynomials, edgewise subdivisions.

1 Introduction

The main objective of this paper is to confirm an Athanasiadis' conjecture on the real-rootedness of the following polynomials arising from the theory of subdivisions:

$$E_r(x + x^2 + \cdots + x^{r-1})^n, \quad (1)$$

where r is a positive integer and E_r is a linear operator on polynomials defined as $E_r(x^n)$ is $x^{n/r}$ if r divides n , 0 otherwise.

The polynomials given in (1) can be explained as local h -polynomials of the r th edgewise subdivision of simplexes. Let us first give a review of the local h -polynomials. The notion of local h -polynomials was introduced by Stanley [7] during his study on the face enumeration of subdivisions of simplicial complexes. Given an n -element set V , let 2^V be the abstract simplex consisting of all subsets of the set V and Γ be a homology subdivision of the simplex 2^V . Let $h(\Delta, x)$ be the h -polynomial of a given simplicial complex Δ . The local h -polynomial of Γ is defined as

$$\ell_V(\Gamma, x) = \sum_{F \subset V} (-1)^{n-|F|} h(\Gamma_F, x),$$

where Γ_F is the restriction of Γ to the face $F \in 2^V$. Athanasiadis [1] made an excellent survey on this topic. He [1, Section 4] also gave several interesting examples of local h -polynomials for certain families of subdivisions of complexes and asked whether these polynomials have only real zeros.

One famous example is about the barycentric subdivision of the simplex 2^V , denoted $\text{sd}(2^V)$. Stanley [7, Proposition 2.4] first showed that the local h -polynomial of $\text{sd}(2^V)$ is the generating function of derangements of order n by the excedance statistic, which was first studied by Brenti [3]. The real-rootedness of these polynomials was proved by Zhang [10].

This paper is concerned about another class of these polynomials, which is the r th edgewise subdivision of 2^V . Given a simplicial complex Δ , its r th edgewise subdivision, denoted $\Delta^{(r)}$, is obtained by subdividing each face $F \in \Delta$ into $r^{\dim(F)}$ faces of the same dimension. Athanasiadis [1, Theorem 4.6] showed that the polynomial $\ell_V((2^V)^{(r)}, x)$ has a nice and surprising expression as shown in (1). This polynomial can also be explained combinatorially as the generating function of Smirnov words [5] by the ascent statistic; see [1, Theorem 4.6] for more details. In this paper, we shall give a positive answer to the real-rootedness of this class of polynomials.

Theorem 1.1. *For any positive integer r , the local h -polynomial of the r th edgewise subdivision of the complex 2^V has only real zeros.*

Our proof is based on the theory of interlacing polynomials. Recently, this approach has been widely used to prove the real-rootedness of several polynomials arising in combinatorics ([6, 8, 9]).

The rest of the paper is organized as follows. In Section 2, we make some necessary preliminaries on interlacing polynomials. Section 3 is dedicated to our proof of Theorem 1.1. The key ingredient is that we show a sequence of interlacing polynomials, from which Theorem 1.1 can be deduced.

2 Preliminaries

In this section, we review some definitions and theorems from the theory of real-rooted polynomials.

We first introduce the notion of interlacing. Given two real-rooted polynomials $f(x)$ and $g(x)$ with positive leading coefficients, let $\{u_i\}$ and $\{v_j\}$ be the set of zeros of $f(x)$ and $g(x)$, respectively. We say that $g(x)$ *interlaces* $f(x)$, denoted $g(x) \preceq f(x)$, if either $\deg f(x) = \deg g(x) = n$ and

$$v_n \leq u_n \leq v_{n-1} \leq \cdots \leq v_2 \leq u_2 \leq v_1 \leq u_1,$$

or $\deg f(x) = \deg g(x) + 1 = n$ and

$$u_n \leq v_{n-1} \leq \cdots \leq v_2 \leq u_2 \leq v_1 \leq u_1.$$

We say that a sequence of real polynomials $(f_1(x), \dots, f_m(x))$ with positive leading coefficients is called *interlacing* if $f_i(x) \preceq f_j(x)$ for all $1 \leq i < j \leq m$. Following Brändén [2],

let \mathcal{F}_n be the set of all interlacing sequences $(f_i)_{i=1}^n$ of polynomials, and \mathcal{F}_n^+ be the subset of $(f_i)_{i=1}^n \in \mathcal{F}_n$ such that the coefficients of f_i are all nonnegative for all $1 \leq i \leq n$.

One central problem is about an $m \times n$ matrix $G = (G_{ij}(x))$ of polynomials which maps \mathcal{F}_n to \mathcal{F}_m (or \mathcal{F}_n^+ to \mathcal{F}_m^+) via a matrix multiplication as follows:

$$G \cdot (f_1, \dots, f_n)^T = (g_1, \dots, g_m)^T.$$

Fisk [4, Chapter 3] considered this problem and gave some basic results. Brändén gave the following characterization for the case when the polynomials considered have all non-negative coefficients.

Theorem 2.1 ([2, Theorem 8.5]). *Let $G = (G_{ij}(x))$ be an $m \times n$ matrix of polynomials. Then $G : \mathcal{F}_n^+ \rightarrow \mathcal{F}_m^+$ if and only if*

1. $G_{ij}(x)$ has nonnegative coefficients for all $i \in [m]$ and $j \in [n]$, and
2. for all $\lambda, \mu > 0$, $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq m$,

$$(\lambda x + \mu)G_{kj}(x) + G_{\ell j}(x) \preceq (\lambda x + \mu)G_{ki}(x) + G_{\ell i}(x). \quad (2)$$

Following Fisk [4], we say that a matrix $M = (m_{i,j})$ is an NX matrix if its entries are either nonnegative real numbers or positive multiples of x . Fisk gave the following criterion to determine whether an NX matrix preserves the interlacing property.

Theorem 2.2 ([4, Proposition 3.72]). *An NX matrix $M = (m_{i,j})$ preserves interlacing polynomials with nonnegative coefficients if the following conditions are satisfied:*

- (1) All entries that lie in the southwest of a multiple of x are multiples of x .
- (2) For any two by two sub-matrix of M having the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or $\begin{pmatrix} ax & bx \\ cx & dx \end{pmatrix}$, we have $ad - bc \geq 0$.
- (3) For any two by two sub-matrix of M having the form $\begin{pmatrix} a & b \\ cx & dx \end{pmatrix}$ or $\begin{pmatrix} ax & b \\ cx & d \end{pmatrix}$, we have $ad - bc \leq 0$.

Before concluding this section, we shall use Theorem 2.1 to give a proof of Theorem 2.2.

Proof. By Theorem 2.1, it suffices to check all the two by two sub-matrices of the NX matrix M . Now the possible 2×2 sub-matrices of M are listed as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ cx & d \end{pmatrix}, \begin{pmatrix} ax & b \\ cx & d \end{pmatrix}, \begin{pmatrix} a & b \\ cx & dx \end{pmatrix}, \begin{pmatrix} ax & b \\ cx & dx \end{pmatrix} \text{ and } \begin{pmatrix} ax & bx \\ cx & dx \end{pmatrix}.$$

We next take the first matrix as an example and all the other matrices can be treated similarly. We need to check for all $\lambda, \mu > 0$ the following interlacing relation is satisfied,

$$b\lambda x + (b\mu + d) \preceq a\lambda x + (a\mu + c).$$

Equivalently,

$$-\frac{b\mu + d}{b\lambda} \leq -\frac{a\mu + c}{a\lambda},$$

which can be reduced to the original condition

$$ad \geq bc.$$

This completes the proof. \square

3 Proof of Theorem 1.1

In the section, we shall present our proof of Theorem 1.1. We first introduce a sequence of polynomials and prove that these polynomials are interlacing (3.2). We then show that one of these polynomials is just the polynomial (1).

Lemma 3.1. *Let $r \geq 2$ be a positive integer. If the sequence $(f_{r-1}(x), \dots, f_1(x), f_0(x))$ of polynomials with nonnegative coefficients is interlacing, and another polynomial sequence $(g_{r-1}(x), \dots, g_1(x), g_0(x))$ is defined as*

$$\begin{aligned} (1 + x + \dots + x^{r-2}) (f_0(x^r) + x f_1(x^r) + \dots + x^{r-1} f_{r-1}(x^r)) \\ = g_0(x^r) + x g_1(x^r) + \dots + x^{r-1} g_{r-1}(x^r), \end{aligned} \quad (3)$$

then the sequence $(g_{r-1}(x), \dots, g_1(x), g_0(x))$ is also interlacing.

Proof. We first write the left hand side of (3) into the following form:

$$\begin{aligned} & 1 (f_0(x^r) + 0 \cdot f_1(x^r) + x^r f_2(x^r) + x^r f_3(x^r) + \dots + x^r f_{r-1}(x^r)) \\ & + x (f_0(x^r) + f_1(x^r) + 0 \cdot f_2(x^r) + x^r f_3(x^r) + \dots + x^r f_{r-1}(x^r)) \\ & + x^2 (f_0(x^r) + f_1(x^r) + f_2(x^r) + 0 \cdot f_3(x^r) + \dots + x^r f_{r-1}(x^r)) \\ & + \dots \\ & + x^{r-1} (0 \cdot f_0(x^r) + f_1(x^r) + f_2(x^r) + f_3(x^r) + \dots + f_{r-1}(x^r)). \end{aligned}$$

Combining it with the right hand side of (3), we obtain the following matrix identity:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 1 \\ x & 0 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ x & x & x & \dots & 1 & 1 & 1 \\ x & x & x & \dots & 0 & 1 & 1 \\ x & x & x & \dots & x & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{r-1}(x) \\ f_{r-2}(x) \\ f_{r-3}(x) \\ \vdots \\ f_2(x) \\ f_1(x) \\ f_0(x) \end{pmatrix} = \begin{pmatrix} g_{r-1}(x) \\ g_{r-2}(x) \\ g_{r-3}(x) \\ \vdots \\ g_2(x) \\ g_1(x) \\ g_0(x) \end{pmatrix}.$$

A routine computation shows that the first matrix in the above formula satisfies the conditions of Theorem 2.2, and hence preserves interlacing polynomials. Hence, since the sequence $(f_{r-1}(x), \dots, f_1(x), f_0(x))$ is interlacing, it follows that so is $(g_{r-1}(x), \dots, g_0(x))$. This completes the proof. \square

By iteratively using the above theorem, we can obtain the following result.

Theorem 3.2. *Suppose that*

$$(1 + x + x^2 + \dots + x^{r-2})^n = h_0(x^r) + xh_1(x^r) + \dots + x^{r-1}h_{r-1}(x^r). \quad (4)$$

Then the sequence polynomial $(h_{r-1}(x), \dots, h_1(x), h_0(x))$ is interlacing.

Now it is time to show our proof of Theorem 1.1.

Proof of Theorem 1.1. An observation is that the polynomial $\ell_V((2^V)^{\langle r \rangle}, x)$ as in (1) is just some $h_j(x)$ in (4) except a power of x . Therefore, by Theorem 3.2, the polynomial $\ell_V((2^V)^{\langle r \rangle}, x)$ is real-rooted, which completes the proof of Theorem 1.1. \square

Acknowledgments. This work was supported by the 973 Project and the National Science Foundation of China.

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